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NODAL SETS FOR SOLUTIONS OF ELLIPTIC EQUATIONS

ROBERT HARDT & LEON SIMON

Here we study, on a connected domain $\Omega \subset \mathbf{R}^n$, the zero set $u^{-1}\{0\}$ of a solution u of an elliptic equation

$$a_{ij}D_iD_ju + b_jD_ju + cu = 0,$$

where a_{ij}, b_j, c are bounded and a_{ij} is continuous.

Our principal result (precisely stated in Theorem (1.7) below) is that the $(n-1)$-dimensional Hausdorff measure of $u^{-1}\{0\}$ is finite in a neighborhood of any point $x_0 \in \Omega$ at which u has finite order of vanishing. (For Lipschitz a_{ij} this holds at *each* point $x_0 \in \Omega$ by the unique continuation theory for elliptic equations.) We actually obtain an explicit bound on the Hausdorff measure of $u^{-1}\{0\}$ in terms of the order of vanishing of u, the modulus of continuity of a_{ij}, and the bounds on a_{ij}, b_j, c.

Notice that in the case the coefficients a_{ij}, b_j, c are analytic, u is then real analytic [8], and the finiteness of the $(n-1)$-dimensional Hausdorff measure of $u^{-1}\{0\}$ is automatic [3, 3.4.8]. The explicit bound on the $(n-1)$-dimensional Hausdorff measure is nevertheless of interest in this case, but a more precise estimate for the real analytic case was already established in [2].

We also show here (in Theorem (1.10)) that if the coefficients are sufficiently smooth then $u^{-1}\{0\}$ decomposes into a disjoint union of the embedded C^1 submanifold $u^{-1}\{0\} \cap \{|Du| > 0\}$ together with the closed set $u^{-1}\{0\} \cap |Du|^{-1}\{0\}$, which we show is countably $(n-2)$-rectifiable. L. Caffarelli and A. Friedman showed already in [1] that $\dim u^{-1}\{0\} \cap |Du|^{-1}\{0\} \le n-2$ in the case of equations of the special form $\Delta u + f(x,u) = 0$. We thank F. H. Lin for pointing out this reference.

In §5 of the present paper we apply the main estimates of §1 and an estimate of Donnelly and Fefferman [2] for the order of vanishing of eigenfunctions to give an asymptotic bound of the $(n-1)$-dimensional measure

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of $u_j^{-1}\{0\}$, where u_j is an eigenfunction corresponding to the jth eigenvalue of the Laplacian on a compact Riemannian manifold. In the real analytic case, a more precise estimate was obtained in [2], but the results of the present paper seem to be the first estimates for the smooth case.

1. Statement of main results

We consider the second order linear equation

$$a_{ij}D_iD_ju + b_jD_ju + cu = 0 \tag{1.1}$$

on a domain $\Omega \subset \mathbf{R}^n$, and we assume the following:

$$a_{ij}(x_0) = \delta_{ij}, \tag{1.2}$$

where x_0 denotes a given point of Ω (note that this assumption really involves no loss of generality because we can always achieve it with a suitable linear transformation, provided the original equation is at least elliptic at x_0),

$$|a_{ij}(x) - a_{ij}(x_0)| \le \sigma(|x - x_0|), \qquad x \in \Omega, \tag{1.3}$$

where σ is an increasing continuous function on $[0, \infty)$ with $\sigma(0) = 0$ (σ is a modulus of continuity for a_{ij} at x_0),

$$\sup |b_j| \le \mu_1, \qquad \sup |c| \le \mu_2, \tag{1.4}$$

where μ_1, μ_2 are constants.

Concerning the solution u of (1.1) we assume $u \in C^1(\Omega) \cap H^{2,2}_{\mathrm{loc}}(\Omega)$ (hence $u \in C^{1,\alpha}(B_\rho(x_0)) \cap H^{2,p}(B_\rho(x_0))$ $\forall \alpha \in (0,1)$, $p \ge 2$, by the elliptic regularity theorem, provided $\rho < \operatorname{dist}(x_0, \partial\Omega)$ and $\sigma(\rho) \le c^{-1}$ for suitable $c = c(n)$), and we consider a point $x_0 \in u^{-1}\{0\}$ at which u has finite order of vanishing. Thus we assume that there exists an integer $d > 0$ such that

$$\limsup_{\rho \downarrow 0} \rho^{-d} \|u\|_\rho > 0, \tag{$*$}$$

where, here and subsequently, $\|u\|_\rho^2 = \rho^{-n} \int_{B_\rho(x_0)} u^2$. (Recall [6], [4] that such a d exists automatically if the a_{ij} are Lipschitz.) Then there exist arbitrarily small numbers R such that

$$\|u\|_R < 2^{d+1} \|u\|_{R/2}, \tag{1.5}$$

because otherwise $\|u\|_\rho \ge 2^{d+1}\|u\|_{\rho/2}$ for all sufficiently small ρ, and iteration of this implies that $\limsup_{\rho\downarrow 0} \rho^{-d-1}\|u\|_\rho < \infty$, contrary to the hypothesis $(*)$.

We subsequently use the notation

$$\delta(\rho) \equiv \sigma(\rho) + \mu_1 \rho + \mu_2 \rho^2. \tag{1.6}$$

Our first result asserts that the $\mathscr{H}^{n-1}$ measure of $u^{-1}\{0\}$ is bounded in a neighborhood of x_0 in terms of n, μ, d, and a certain constant $\rho_0 \in (0, R)$. Specifically we have the following.

(1.7) Theorem. *There exist constants $c = c(n) > 0$ and $\varepsilon_0 = \varepsilon_0(n) \in (0, 1/2]$ such that if $x_0 \in u^{-1}\{0\}$, if $\rho_0 > 0$ is small enough to ensure that $\delta(R) \leq \varepsilon^{3d}$ and $B_R(x_0) \subset \Omega$, with $R = \varepsilon^{-1}\rho_0$ and $\varepsilon = \varepsilon_0/d^{2n+3}$, and if* (1.1)–(1.5) *all hold, then*

(i)

$$\mathscr{H}^{n-1}(B_\rho(x_0) \cap u^{-1}\{0\}) \leq c d \rho^{n-1} \; (< \infty) \quad \forall \rho \leq \rho_0,$$

(ii)

$$\dim\{B_{\rho_0}(x_0) \cap u^{-1}\{0\} \cap |Du|^{-1}\{0\}\} \leq n - 2.$$

Thus $B_\rho(x_0) \cap u^{-1}\{0\}$ decomposes into a union of the $(n-1)$-dimensional C^1 submanifold $B_\rho(x_0) \cap u^{-1}\{0\} \cap \{|Du| > 0\}$ with finite $(n-1)$-dimensional measure, and a closed set $B_\rho(x_0) \cap u^{-1}\{0\} \cap \{|Du|^{-1}\{0\}$ of dimension $\leq n - 2$.

(1.8) Remarks. (1) An inequality like (ii) was established for equations of the form $\Delta u + f(x, u) = 0$, in case Δu is the standard Euclidean Laplacian, in [1]. (See the discussion in (1.9) below.)

(2) It is perhaps worth mentioning explicitly that in the course of the proof of Theorem (1.7) (see Remark (4.6) below) we show that for any given $\theta > 0$ we can bound the order of vanishing of u at any $y \in B_\rho(x_0)$ by $d + \theta$, for suitable $\rho > 0$. Of course, if $u \in C^d(\Omega)$, the order of vanishing of u is trivially $\leq d$ in some ball $B_\rho(x_0)$.

(3) The results of the above theorem remain true (and the proofs need very little modification) in case equation (1.1) is replaced by the divergence-form equation

$$D_i(a_{ij} D_j u) + D_i(b_i u) + \tilde{b}_i D_i u + cu = 0, \tag{1.1'}$$

if

$$|a_{ij}| \leq \gamma, \qquad a_{ij}\xi_i\xi_j \geq |\xi|^2, \tag{1.2'}$$

$$|b_i| + |\tilde{b}_i| + |c| \leq \mu \tag{1.3'}$$

hold, and a_{ij}, b_j are Hölder continuous with exponent α for some $\alpha \in (0, 1)$. Actually it would suffice in (1.3′) that $\tilde{b}_i, c \in L^p$ for suitable p.

(4) Of course the results of Theorem (1.7) apply to fully nonlinear second order elliptic equations of the form

$$a_{ij}(x, u, Du, D^2u) D_i D_j u + b_j(x, u, Du, D^2u) D_j u + c(x, u, Du, D^2u) u = 0,$$

provided $u \in C^2$, a_{ij} is continuous, and b_j, c are bounded, because such an equation has the form (1.1)–(1.4) with suitable σ, μ.

Next we want to give a more precise discussion of the set $u^{-1}\{0\} \cap |Du|^{-1}\{0\}$ near x_0. For this we need $a_{ij}, b_j, c \in C^d(\Omega)$, so that $u \in C^{d+1,\alpha}(\Omega)$ $\forall \alpha \in (0,1)$ by the elliptic regularity theory. Then we have:

(1.9) Lemma. *If* (1.1) *holds with* $u \not\equiv$ const, $a_{ij}, b_j, c \in C^d(\Omega)$, a_{ij} *is positive definite on* Ω, *and* $(*)$ *holds at each point* $x_0 \in u^{-1}\{0\}$, *then* $u^{-1}\{0\} \cap |Du|^{-1}\{0\}$ *decomposes into the countable union of subsets of a pairwise disjoint collection of smooth* $(n-2)$*-dimensional submanifolds.* (*Thus* $u^{-1}\{0\} \cap |Du|^{-1}\{0\}$ *is a countably* $(n-2)$*-rectifiable subset in the terminology of* [3].)

Proof. The argument is essentially that used by Caffarelli and Friedman [1]. For each $q = 1, 2, 3, \cdots$, we let

(1) $$S_q = \{x \colon D^\alpha u(x) = 0 \ \forall |\alpha| \le q \text{ and } D^{q+1}u(x) \ne 0\}.$$

Evidently, in view of Remark (1.8)(3) we have, for any x_0 with $u(x_0) = 0$, $Du(x_0) = 0$ and for suitable $\rho > 0$, that

(2) $$B_\rho(x_0) \cap \{x \colon u(x) = 0, Du(x) = 0\} = \left(\cup_{q=1}^d S_q\right) \cap B_\rho(x_0),$$

and of course

(3) $$S_p \cap S_q = \emptyset \quad \text{for } p \ne q.$$

Now consider $x \in S_q$ and choose a multi-index β with $|\beta| = q-1$ and $\operatorname{Hess}(D^\beta u)(x) \ne 0$. By applying D^β to each side of (1.1) and using the fact that $D^\alpha u = 0$ $\forall \alpha$ with $|\alpha| \le q$, we get

$$a_{ij}(x) \frac{\partial^2 (D^\beta u)}{\partial x^i \partial x^j} = 0,$$

so that, since $(a_{ij}(x))$ is positive definite and $\operatorname{Hess}(D^\beta u)(x) \ne 0$, we must have $\operatorname{rank} \operatorname{Hess}(D^\beta u(x)) \ge 2$.

Thus we can choose $i_1, i_2 \in \{1, \cdots, n\}$ such that $\operatorname{grad}(D_{i_1} D^\beta u)$, $\operatorname{grad}(D_{i_2} D^\beta u)$ are linearly independent at x, hence for some $\rho > 0$

$$B_\rho(x) \cap (D_{i_1} D^\beta u)^{-1}\{0\} \cap (D_{i_2} D^\beta u)^{-1}\{0\}$$

is an embedded $(n-2)$-dimensional submanifold $M_{\beta,x,q}$ which contains all of $B_\rho(x) \cap S_q$.

We have thus shown that for each $x \in u^{-1}\{0\} \cap |Du|^{-1}\{0\}$ we can find $\rho > 0$ and smooth embedded $(n-2)$-dimensional submanifolds $M_{\beta_1,x,q_1}, \cdots, M_{\beta_r,x,q_r}$ such that

$$B_\rho(x) \cap u^{-1}\{0\} \cap |Du|^{-1}\{0\} \subset \cup_{j=1}^r M_{\beta_j,x,q_j}.$$

This completes the proof of Lemma (1.9).

Notice that by combining Theorem (1.7), Lemma (1.9) and the unique continuation theory for elliptic equations we arrive at:

(1.10) Theorem. *Suppose that* (1.1) *holds with* $u \not\equiv$ const *and that in addition* $a_{ij}, b_j, c \in C^\infty(\Omega)$ *and* a_{ij} *is positive definite on* Ω. *Then* $u^{-1}\{0\}$ *decomposes into the disjoint union* $(u^{-1}\{0\} \cap \{|Du| > 0\}) \cup (u^{-1}\{0\} \cap |Du|^{-1}\{0\})$ *of smooth* $(n-1)$*-dimensional manifold having finite* $(n-1)$*-dimensional measure in each compact subset of* Ω *and a closed countably* $(n-2)$*-rectifiable subset.*

2. An estimate for the zero set of a polynomial

(2.1) Theorem. *Let* $q: \mathbf{R}^n \to \mathbf{R}$ *be a polynomial of degree* $\le d$ *and suppose that* $\dim q^{-1}\{0\} \le k$. *Then*

$$\mathscr{H}^k(q^{-1}\{0\} \cap B_1) \le cd^{n-k},$$

where c depends only on n.

Proof. In case $k = 0$, $q^{-1}\{0\}$ is a finite set, and the inequality follows from [7, Theorem 2], which bounds the sum of Betti numbers, hence the number of components of $q^{-1}\{0\}$, by $d(2d-1)^{n-1} \le cd^n$.

In case $k > 0$, we use the coordinate projections

$$p_\lambda: \mathbf{R}^n \to \mathbf{R}^k, \qquad p_\lambda(x^1, \cdots, x^n) = (x^{\lambda_1}, \cdots, x^{\lambda_k}),$$

defined for $\lambda \in \Lambda(n,k) \equiv \{(i_1, \cdots, i_k) \in \mathbf{Z}^k: q \le i_1 < \cdots < i_k \le n\}$. Assuming $\dim q^{-1}\{0\} \le k$, we infer from the inequality [3, 2.10.28] that $q^{-1}\{0\} \cap p_\lambda^{-1}\{y\}$ is finite for each $\lambda \in \Lambda(n,k)$ and almost all $y \in \mathbf{R}^k$. For such λ, y,

$$\operatorname{card} q^{-1}\{0\} \cap p_\lambda^{-1}\{y\} \le d(2d-1)^{n-k-1} \le cd^{n-k}$$

as in the previous case because $q^{-1}\{0\} \cap p_\lambda^{-1}\{y\}$ is defined by the vanishing of a polynomial of degree $\le d$ on a Euclidean space of dimension $n-k$. Using this estimate and [3, 3.2.27] we conclude

$$\mathscr{H}^k(q^{-1}\{0\} \cap B_1) \le \sum_{\lambda \in \Lambda(n,k)} \int_{p_\lambda(B_1)} \operatorname{card} q^{-1}\{0\} \cap p_\lambda^{-1}\{y\}\, d\mathscr{L}^k y \le cd^{n-k}.$$

3. Estimates for the zero sets of harmonic polynomials

Let ϕ be a harmonic polynomial of degree d in $\mathbf{R}^n$ and $\phi \not\equiv$ const. We note that (2.1) (with $k = n-1$) gives

$$\mathscr{H}^{n-1}(\phi^{-1}\{0\} \cap B_1) \le cd. \tag{3.1}$$

Also, $\dim |D\phi|^{-1}\{0\} \le n-2$, because otherwise we would have $\dim |D\phi|^{-1}\{0\} = n-1$ and by stratification of algebraic varieties (see [11]) we could find a smooth connected $(n-1)$-dimensional submanifold M of $\mathbf{R}^n$ with $|D\phi| \equiv 0$ on M. Since $\phi \equiv$ const on M we would then have $\phi -$ const satisfying zero Cauchy data on M, thus contradicting unique continuation for harmonic functions. In particular, we can now deduce from Theorem (2.1) (with $k = n-2$) that

$$\mathscr{H}^{n-2}(|D\phi|^{-1}\{0\} \cap B_1) \le cd^2. \tag{3.2}$$

Our main estimate for harmonic polynomials is given in the following theorem.

(3.3) Theorem. *There are constants $\theta = \theta(n) \in (0, 1/2)$ and $c = c(n) > 0$ such that if ϕ is a harmonic polynomial of degree d in $\mathbf{R}^n$, $\sup_{B_1} |\phi - \phi(0)| = 1$, and $|D\phi(0)| \le (\theta\varepsilon)^{d-1}$, then*

$$\mathscr{L}^n(B_1 \cap \{x\colon \operatorname{dist}(x, \{|D\phi| \le (\theta\varepsilon)^{d-1}\}) < \varepsilon\}) \le cd^{2n+2}\varepsilon^2 \log \varepsilon^{-1}$$

for each $\varepsilon \in (0, 1/2]$.

Remark. It seems likely that the lemma may be true without the factor $\log \varepsilon^{-1}$ on the right, but such an inequality would not significantly improve the main results of the present paper.

Proof of Theorem (3.3). The proof is a fairly straightforward application of Theorem (2.1) together with standard estimates for harmonic functions and the coarea formula.

First notice that we must automatically have that d is ≥ 2 and that for each $z \in B_1(0)$ and each $\varepsilon \in (0, 1/2]$

$$\sup_{B_\varepsilon(z)} |D^2\phi| \ge \theta(2\theta\varepsilon)^{d-2} \tag{1}$$

for suitable $\theta = \theta(n) \in (0, 1/2)$. Indeed the given facts about ϕ tell us that

$$\sup_{B_1(0)} |D^2\phi| \ge c^{-1}, \qquad c = c(n), \tag{2}$$

so $d \ge 2$. Also if (1) were false for a given $z \in B_1(0)$, then standard estimates for the derivatives of harmonic functions would imply that $(\alpha!)^{-1}|D^\alpha\phi(z)| \le c\theta(c\theta)^{d-2}$ for every multi-index α with $2 \le |\alpha| \le d$, which for small enough $\theta = \theta(n)$ contradicts (2) and the fact that each component of $D^2\phi$ is a polynomial of degree $\le d-2$.

We now fix $\theta = \theta(n)$ so that (1) holds, and we proceed to prove the theorem.

Take any $y \in B_1(0)$ with $|D\phi(y)| \le (\theta\varepsilon)^{d-1}$. Since each component of $D^2\phi$ behaves like a *homogeneous* polynomial of degree $\le d-2$ near infinity, the growth estimates of the appendix, together with standard estimates for the maximum of a harmonic function in terms of its L^2 norm (over a larger set), imply

$$(3)\qquad \varepsilon \sup_{B_{(1+\sigma)\varepsilon}(y)} |D^3\phi| + \sup_{B_{(1+\sigma)\varepsilon}(y)} |D^2\phi| \le c\sigma^{-1}(1+3\sigma)^d \sup_{B_{(1-\sigma)\varepsilon}(y)} |D^2\phi| \le c\sigma^{-1}(1+3\sigma)^d \sup_{B_\varepsilon(y)} |D^2\phi|$$

for $\sigma \in (0, 1/4]$, where $c = c(n)$.

Notice also that if $\beta \in (0,1)$, and $x_1 \in \overline{B}_\varepsilon(y)$ is such that $|D^2\phi(x_1)| = \sup_{B_\varepsilon(y)} |D^2\phi|$, then by (3) we get

$$(4)\qquad |D^2\phi(x) - D^2\phi(x_1)| \le c\beta\sigma^{-1}(1+3\sigma)^d |D^2\phi(x_1)|, \qquad x \in B_{\beta\varepsilon}(x_1),$$

so that

$$c\beta\sigma^{-1}(1+3\sigma)^d \le 1/2 \Rightarrow |D^2\phi(x_1)| \le 2 \inf_{B_{\beta\varepsilon}(x_1)} |D^2\phi|.$$

Thus if we select

$$(5)\qquad \sigma = c^{-1}d^{-1}, \qquad \beta = \gamma d^{-1}$$

for suitable $c = c(n)$ and $\gamma = \gamma(n)$, then

$$(6)\qquad |D^2\phi(x_1)| \le 2 \inf_{B_{\gamma\varepsilon/d}(x_1)} |D^2\phi|.$$

Also if $x \in B_\varepsilon(y)$, then trivially for any unit vector $\tau \in \mathbf{R}^n$

$$|D_\tau\phi(x) - D_\tau\phi(y)| \le \varepsilon |D^2\phi(x_1)|,$$

and hence

$$\begin{aligned} |D_\tau\phi(x)| &\le (\theta\varepsilon)^{d-1} + \varepsilon|D^2\phi(x_1)| \\ &\le c\varepsilon|D^2\phi(x_1)) \quad \text{by (1)} \\ &\le c\varepsilon \inf_{B_{\gamma\varepsilon/d}(x_1)} |D^2\phi| \quad \text{by (6)}, \end{aligned}$$

so that

$$(7)\qquad |D_\tau\phi(x)| \le c\varepsilon|D^2\phi(x)| \quad \text{at each point } x \in B_{\gamma\varepsilon/d}(x_1),$$

where $\gamma = \gamma(n)$ (as in (5)).

By (4) and (6) we also conclude that if τ_1, τ_2 are any pair of unit vectors in $\mathbf{R}^n$, and $x \in B_{\gamma\varepsilon/d}(x_1)$, then

$$(8)\qquad |D_{\tau_1}D_{\tau_2}\phi(x) - D_{\tau_1}D_{\tau_2}\phi(x_1)| \le c\gamma \inf_{B_{\gamma\varepsilon/d}(x_1)} |D^2\phi|.$$

Next, let $\mathscr{O}_\gamma$ be a collection of orthonormal bases of $\mathbf{R}^n$ such that for any orthonormal basis $\{\tau_1, \cdots, \tau_n\}$ of $\mathbf{R}^n$ there is an orthonormal basis $\{\tilde{\tau}_1, \cdots, \tilde{\tau}_n\} \in \mathscr{O}_\gamma$ with

$$(9) \qquad |\tau_j - \tilde{\tau}_j| < \gamma/d, \qquad j = 1, \cdots, n.$$

We can, and we shall, choose such a set $\mathscr{O}_\gamma$ in such a way that

$$(10) \qquad \text{number of elements in } \mathscr{O}_\gamma \le c\gamma^{1-n} d^{n-1}, \qquad c = c(n).$$

With $x_1 \in \overline{B}_\varepsilon(y)$ as in (4) above, let $\tau_1, \cdots, \tau_n$ be an orthonormal basis of $\mathbf{R}^n$ such that

$$|D_{\tau_1} D_{\tau_1} \phi(x_1)| = \max_{j \in \{1, \cdots, n\}} |D_{\tau_j} D_{\tau_j} \phi(x_1)|,$$
$$|D_{\tau_2} D_{\tau_2} \phi(x_1)| = \max_{j \in \{2, \cdots, n\}} |D_{\tau_j} D_{\tau_j} \phi(x_1)|,$$
$$D_{\tau_i} D_{\tau_j} \phi(x_1) = 0, \qquad i \ne j.$$

Furthermore since $\sum_{j=1}^n D_{\tau_j} D_{\tau_j} \phi \equiv 0$, we have

$$|D_{\tau_2} D_{\tau_2} \phi(x_1)| \ge (n-1)^{-1} |D_{\tau_1} D_{\tau_1} \phi(x_1)|.$$

Then by (8) and (9) we can select an orthonormal basis $\tilde{\tau}_1, \cdots, \tilde{\tau}_n \in \mathscr{O}_\gamma$ such that

$$(11) \qquad \begin{aligned} |\phi_{11}(x)| &\ge c^{-1} \max_{j \in \{1, \cdots, n\}} |\phi_{jj}(x)|, \\ |\phi_{22}(x)| &\ge c^{-1} \max_{j \in \{1, \cdots, n\}} |\phi_{jj}(x)|, \\ |\phi_{ij}(x)| &\le c\gamma |\phi_{11}(x)|, \qquad i \ne j, \end{aligned}$$

for each $x \in B_{\gamma\varepsilon/d}(x_1)$, where we use the notation $\phi_{ij}(x) \equiv D_{\tilde{\tau}_i} D_{\tilde{\tau}_j} \phi(x)$.

Now by (6), (7), and (11) we have, with $\phi_j = D_{\tilde{\tau}_j} \phi$,

$$(12) \qquad \begin{aligned} c^{-1} &\le \varepsilon^2 |D\phi(x)|^{-2} |D^2 \phi|^2 \\ &\le c\varepsilon^2 ((\phi_1(x))^2 + (\phi_2)^2)^{-1} \\ &\quad \times (|D\phi_1|^2(x) |D\phi_2|^2(x) - (D\phi_1(x) \cdot D\phi_2(x))^2)^{1/2} \end{aligned}$$

for each $x \in B_{\gamma\varepsilon/d}(x_1)$, provided we take $\gamma = \gamma(n)$ sufficiently small.

Define J to be the Jacobian of the transformation $x \mapsto (\phi_1(x), \phi_2(x))$, $x \in B_{\gamma\varepsilon/d}(x_1)$, that is, $J = \sqrt{|D\phi_1|^2 |D\phi_2|^2 - (D\phi_1 \cdot D\phi_2)^2}$. In view of (12) we have

$$(13) \qquad c^{-1} \le \varepsilon^2 (\phi_1^2(x) + \phi_2^2(x))^{-1} J(x) \quad \text{at each point } x \in B_{\gamma\varepsilon/d}(x_1).$$

Also by (11), (6), and (1) we have $|\phi_{22}(x)|, |\phi_{11}(x)| \ge (\theta\varepsilon)^{d-2}$ for $x \in B_{\gamma\varepsilon/d}(x_1)$, and hence

$$(14) \qquad |\phi_2|, |\phi_1| \ge c^{-1} \gamma d^{-1} (\theta\varepsilon)^{d-1}$$

on a subset A of $B_{\gamma\varepsilon/d}(x_2)$ with $\mathscr{L}^n(A) \geq c^{-1}(\gamma\varepsilon/d)^n$, where $c = c(n)$.

By (13) and (14) we deduce from the coarea formula [3, 3.2.22] that

$$\begin{aligned} c^{-1}d^{-n}\varepsilon^n \leq \varepsilon^2 \int_{(c^{-1}\varepsilon)^d}^{c^d} \int_{(c^{-1}\varepsilon)^d}^{c^d} (s^2+t^2)^{-1} \\ \times \sum_{\mathscr{O}_\gamma} \mathscr{H}^{n-2}(\{x \in B_\varepsilon(y) : \phi_1(x) = s, \phi_2(x) = t\})\, ds\, dt, \end{aligned} \tag{15}$$

for suitable $c = c(n)$.

Now select a maximal pairwise-disjoint collection of balls $\{B_{\varepsilon/2}(y_j)\}_{j=1,\cdots,M}$ with $y_j \in \{x \in B_1(0) : |D\phi(x)| \leq (\theta\varepsilon)^{d-1}\}$, and sum over j after replacing y by y_j in (15). Then (keeping in mind that $\mathscr{O}_\gamma$ has $\leq cd^{n-1}$ elements) we get

$$\begin{aligned} &\mathscr{L}^n(\{x \in B_1(0) : \operatorname{dist}(x, \{|D\phi| \leq (\theta\varepsilon)^{d-1}\}) < \varepsilon\}) \\ &\quad \leq cd^{2n-1}\varepsilon^2 \int_{(c^{-1}\varepsilon)^d}^{c^d} \int_{(c^{-1}\varepsilon)^d}^{c^d} (s^2+t^2)^{-1} \\ &\quad \cdot \max_{\mathscr{O}_\gamma} \mathscr{H}^{n-2}(\{x \in B_2(0) : (\phi_1(x)-s)^2 + (\phi_2(x)-t)^2 = 0\})\, ds\, dt. \end{aligned}$$

The required inequality now follows from (10) and Theorem (2.1) because $\dim\{x \in B_2(0) : \phi_1(x) = s, \phi_2(x) = t\} \leq n-2$ for a.e. $(s,t) \in (0,1) \times (0,1)$ by the coarea formula, and because $\int_\alpha^{\alpha^{-1}} \int_\alpha^{\alpha^{-1}} (s^2+t^2)^{-1}\, ds\, dt \leq c \log \alpha^{-1}$ for each $\alpha \in (0, 1/2)$.

4. Proof of Theorem (1.7)

Notice that by translation $x \mapsto x - x_0$ and a homothety $x \mapsto \rho_0^{-1}x$, so that $B_R(x_0)$ is transformed to $B_{R_1}(0)$, $R_1 = \varepsilon^{-1}$, the equation is transformed into an equation of the form

$$\Delta u = a \cdot D^2 u + b \cdot Du + cu \quad \text{on } B_{R_1}(0) \tag{4.1}$$

with

$$\sup_{B_{R_1}}(|a| + |b| + |c|) \leq \delta(\varepsilon^{-1}\rho_0), \tag{4.2}$$

where δ is as in (1.6).

We use the notation that if $B = B_\rho(y) \subset B_{R_1}(0)$, then

$$\|u\|_{y,\rho} = \left(\rho^{-n}\int_B u^2\right)^{1/2}.$$

Of course then by (1.5) we have

$$(4.3) \qquad 0 < \|u\|_{0,R_1} \le 2^d \|u\|_{0,R_1/2}.$$

We assume

$$(4.4) \qquad \delta(\varepsilon^{-1}\rho_0) \le \varepsilon^{3d},$$

where $\varepsilon > 0$ is to be chosen. Then the growth estimates of the appendix give that we can select $\varepsilon_0 = \varepsilon_0(n) > 0$ such that if $\varepsilon \le \varepsilon_0$ then

$$(4.5) \qquad 0 < \|u\|_{y,\rho} \le 2^{2d+1/2}\|u\|_{y,\rho/2}$$

for every $\rho \le R_1 - 1$ and $y \in B_1(0)$. Indeed it would suffice merely that $\delta(\varepsilon^{-1}\rho_0) \le \varepsilon^d$ for suitable $\varepsilon = \varepsilon(n)$ for this.

(4.6) Remark. Notice this implies that the order of vanishing of u at any such y is $\le 2d + 1/2$, because by iteration it gives

$$\|u\|_{y,\rho/2^j} \ge 2^{-j(2d+1/2)}\|u\|_{y,\rho}.$$

By only a slightly more complicated argument (still based directly on (*) of §1 and the growth estimates of the appendix), we can show that the order of vanishing of u at such y is $\le d + \theta$ for any given $\theta > 0$, provided $y \in B_{\rho_1}(0)$ with ρ_1 small enough, depending on θ, d.

We shall need the following lemma concerning approximation by harmonic polynomials.

(4.7) Lemma. *There is $\varepsilon_0 = \varepsilon_0(n)$ such that if $\varepsilon \le \varepsilon_0$,* (4.1)–(4.4) *hold, and $B = B_\rho(y)$ with $\rho \le 1$ and $|y| \le 1$, then there is a harmonic polynomial ϕ^B of degree $\le 3d$ such that*

$$|u_B - \phi^B|_{C^1(B_2(0))} \le (c\varepsilon)^d,$$

where u_B is defined by $u_B(x) = \|u\|_{y,\rho}^{-1} u(y + \rho x)$, and $c = c(n)$.

Remark. A similar result holds (by essentially the same argument) with ϕ^B a degree d harmonic polynomial, provided we are willing to assume the stronger condition $\delta(\varepsilon^{-d}\rho_0) \le \varepsilon^{5d^2}$ in place of (4.4).

Proof of Lemma (4.7). $v = u_B$ satisfies an equation of the form

$$(1) \qquad \Delta v = \tilde{a} \cdot D^2 v + \tilde{b} \cdot Dv + \tilde{c} v$$

on $B_{R_1}(0)$, with $|\tilde{a}| + |\tilde{b}| + |\tilde{c}| \le \delta(\varepsilon^{-1}\rho_0)$ and $R_1 = \varepsilon^{-1}$. By definition of u_B and by (4.5) we have

$$(2) \qquad \|u_B\|_{0,1} = 1, \qquad \|u_B\|_{0,R_1-1} \le (cR_1)^{2d+1/2}.$$

Let ψ be the harmonic function on $B_{R_1/2}$ with $\psi = u$ on $\partial B_{R_1/2}$. Then $\tilde{v} \equiv v - \psi$ satisfies the equation

$$(3) \qquad \Delta \tilde{v} = \tilde{a} \cdot D^2 v + \tilde{b} \cdot Dv + \tilde{c} v.$$

By the $H^{2,p}$ estimates for elliptic equations [5, Theorem 9.11] (applied to both (1) and (3)) and by (2) we see that (for $\varepsilon = \varepsilon(n)$ sufficiently small)

$$(4) \qquad |u_B - \psi|_{C^1(B_{R_1/2}(0))} \le (cR_1)^{2d+1/2}, \qquad c = c(n).$$

By standard estimates for harmonic functions (keeping in mind that $\|\psi\|_{0,R_1/2} \le (cR_1)^{2d+1/2}$ by (2) and (4)), we conclude

$$(5) \qquad |\psi - \psi_d|_{C^1(B_2(0))} \le c^d R_1^{-d},$$

where ψ_d denote the terms in the Taylor series expansion of ψ about 0 up to and including terms of degree $3d$.

It then follows that

$$\begin{aligned}|u_B - \psi_d|_{C^1(B_2(0))} &\le |u_B - \psi|_{C^1(B_2(0)} + |\psi - \psi_d|_{C^1(B_2(0))} \\ &\le |u_B - \psi|_{C^1(B_{R_1/2}(0))} + |\psi - \psi_d|_{C^1(B_2(0))} \\ &\le c((c\varepsilon^{-1})^{2d}\delta(\varepsilon^{-1}\rho_0) + (c\varepsilon)^d),\end{aligned}$$

by (4), (5), and the $H^{2,p}$ theory. Thus by (4.4)

$$|u_B - \phi^B|_{C^1(B_2(0))} \le (c\varepsilon)^d,$$

with $\phi^B = \psi_d$.

We shall also use the following "nodal set comparison lemma"; the reader should keep in mind that this is going to be applied with $w_1 = \text{const}\, u_B$, $w_2 = \text{const}\, \phi^B$, with u_B, ϕ^B as in the above Lemma (4.7).

(4.8) Lemma. *There exists $\eta_0 = \eta_0(n) \in (0, 1/2]$ such that with $\eta \in (0, \eta_0)$, if $w_1, w_2 \in C^{1,1/2}(B_2(0))$ with $|w_j|_{C^{1,1/2}} \le 1$, $j = 1, 2$, and if $|w_1 - w_2|_{C^1} < \eta^5/2$, then*

$$\begin{aligned}&\mathscr{H}^{n-1}(B_{2-\eta}(0) \cap \{w_1 = 0, |Dw_1| > \eta\}) \\ &\qquad \le (1 + c\sqrt{\eta})\mathscr{H}^{n-1}(B_2(0) \cap \{w_2 = 0, |Dw_2| > \eta/2\}).\end{aligned}$$

Proof. For small enough η (depending only on n) the following argument is valid; c will denote any constant depending only on n. Let $S_0 = w_1^{-1}\{0\}$, $S_1 = \{w_1 = 0, |Dw_1| > \eta\}$, $S_2 = \{w_2 = 0, |Dw_2| > \eta/2\}$, and take any $x \in \overline{B}_{2-\eta}(0) \cap S_1$. Since $|w_j|_{C^{1,1/2}} \le 1$ and $|w_1 - w_2|_{C^1} \le \eta^5$, we have

$$(1) \qquad \begin{aligned}&|Dw_1| \ge \eta - \eta/5 > 3\eta/4 \quad \text{on } B_{\eta^2/25}(x), \\ &|Dw_2| \ge \eta - \eta/5 - \eta^5 > 3\eta/4 \quad \text{on } B_{\eta^2/25}(x)\end{aligned}$$

and, defining $\nu_j = |Dw_j|^{-1}Dw_j$, we calculate

$$(2) \qquad \begin{aligned}&|\nu_j(y_1) - \nu_j(y_2)| \le c\eta^{-1}|y_1 - y_2|^{1/2} \le c\eta^{1/2}, \\ &\qquad\qquad \text{for } y_1, y_2 \in B_{2\eta^3}(x),\ j = 1, 2,\end{aligned}$$

(3) $|\nu_1(y) - \nu_2(y)| \le c\eta^{-1}|Dw_1(y) - Dw_2(y)| \le c\eta^4, \qquad y \in B_{2\eta^3}(x).$

In particular by (2) and (3) we have

(4) $$|\nu_2(y) - \nu_1(x)| \le c\eta^{1/2}, \qquad y \in B_{2\eta^3}(x).$$

We note that

(5) $$S_2 \cap B_{\eta^4}(x) \ne \varnothing,$$

because, again using $|w_1|_{C^{1,1/2}} \le 1$ together with (1) and the fact that $w_1(x) = 0$,

$$w_1(x + \eta^4\nu_1(x)) > \tfrac{1}{2}\eta^5, \quad w_1(x - \eta^4\nu_1(x)) < -\tfrac{1}{2}\eta^5,$$

and hence, since $|w_1 - w_2|_{C^1} \le \eta^5/2$, we have

$$w_2(x + \eta^4\nu_1(x)) > 0, \quad w_2(x - \eta^4\nu_1(x)) < 0,$$

and hence $w_2(x + \theta\eta^4\nu_1(x)) = 0$ for some $\theta \in (0,1)$, thus establishing (5).

Next, with $x \in S_1 \cap \overline{B}_{2-\eta}(0)$ as above and with T_x denoting the hyperplane containing x and normal to $\nu_1(x)$, we claim

(6)
$$\begin{aligned} S_k \cap B_{\eta^3}(x) &= B_{\eta^3}(x) \cap \operatorname{graph}_{T_x} \psi_x^k \\ &\equiv B_{\eta^3}(x) \cap \{y + \psi_x^k(y)\nu_1(x) : y \in T_x \cap B_{\eta^3}(x)\}, \qquad k = 0, 2, \end{aligned}$$

where $\psi_x^k \in C^1(T_x \cap B_{\eta^3}(x))$ with

(7) $$|D\psi_x^k(y)| \le c\eta^{1/2}, \qquad k = 0, 2.$$

Indeed by (2) and (4) it is at least clear that $S_k \cap B_{2\eta^3}(x)$ is contained in a *union* of such graphs over the larger domain $T_x \cap B_{2\eta^3}(x)$. An elementary argument using (1) and the mean-value theorem for functions of 1 variable then justifies (6) and (7).

Notice that (5) and (7) guarantee that

(8) $$|\psi_x^0(y) - \psi_x^2(y)| \le c\eta^4 + c\eta^3\eta^{1/2} \le c\eta^{7/2}.$$

The required area comparison is now fairly evident from (6), (7), and (8). Specifically, let $\{B_{\eta^3/4}(x_j)\}_{j=1,\cdots,N}$ be a maximal pairwise-disjoint collection of balls with $x_j \in S_1 \cap \overline{B}_{2-\eta}(0)$. Then

(9) $$\{B_{\eta^3/2}(x_j)\}_{j=1,\cdots,N} \text{ covers } S_1 \cap \overline{B}_{2-\eta}(0),$$

and there is a c $(= c(n))$ such that if $\mathscr{F} \subset \{1, \cdots, N\}$, then

(10) $$\cap_{j\in\mathscr{F}} B_{\eta^3}(x_j) \ne \varnothing \Rightarrow \mathscr{F} \text{ has } \le c \text{ elements.}$$

Let $\phi_1, \cdots, \phi_N$ be a partition of unity for $S_1 \cap \overline{B}_{2-\eta}(0)$ with

(11) $\quad \operatorname{support} \phi_j \subset B_{\eta^3}(x_j), \quad \phi_j \geq c^{-1}$ on $B_{\eta^3/2}$, and $|D\phi_j| \leq c/\eta^3$.

Then

$$\begin{aligned}
&\mathscr{H}^{n-1}(S_1 \cap \overline{B}_{2-\eta}(0)) \\
&\quad \leq \sum_{j=1}^{N} \int_{S_0 \cap B_{\eta^3}(x_j)} \phi_j \, d\mathscr{H}^{n-1} = \sum_{j=1}^{N} \int_{T_{x_j} \cap B_{\eta^3}(x_j)} F_j^0(y)\, dy \\
&\quad = \sum_{j=1}^{N} \int_{T_{x_j} \cap B_{\eta^3}(x_j)} F_j^2(y)\, dy + \sum_{j=1}^{N} \int_{T_{x_j} \cap B_{\eta^3}(x_j)} (F_j^0 - F_j^2)(y)\, dy \\
&\quad \leq \mathscr{H}^{n-1}(S_2 \cap B_2(0)) + \sum_{j=1}^{N} \int_{T_{x_j} \cap B_{\eta^3}(x_j)} (F_j^0 - F_j^2)(y)\, dy,
\end{aligned}$$

where (in the notation of (6))

$$F_j^k(y) = \phi_j(y + \psi_{x_j}^k(y)\nu_1(x_j))\sqrt{1 + |D\psi_{x_j}^k(y)|^2}, \qquad k = 0, 2.$$

In view (6), (7), (8), (10), and (11) we conclude that

$$\mathscr{H}^{n-1}(S_1 \cap B_{2-\eta}(0)) \leq (1 + c\sqrt{\eta})\mathscr{H}^{n-1}(S_2 \cap B_2(0)),$$

as required.

We are now going to give the proof of Theorem (1.7). We let $\eta = (c\varepsilon)^{d/5}/2$, with $\varepsilon \leq \varepsilon_0(\eta)$ and $\varepsilon_0(n)$ sufficiently small so that (4.5), (4.7), and (4.8) above can all be applied. Keeping in mind the affine transformation described at the beginning of this section, we see that for (1.7)(i) it suffices to prove

$$\mathscr{H}^{n-1}(u^{-1}\{0\} \cap B_1(0)) \leq cd.$$

First, by Lemma (4.7), for each ball $B = B_\rho(y)$ with $\rho \leq 1$, $y \in u^{-1}\{0\} \cap B_1(0)$, we can find a harmonic polynomial ϕ^B of degree $\leq 3d$ such that

(1) $$|u_B - \phi^B|_{C^1(B_2(0))} \leq \eta^5/2.$$

Since $y \in u^{-1}\{0\}$, by subtracting a suitable constant from ϕ^B, we can also arrange, without upsetting the inequality (1), that $\phi^B(0) = 0$, and, since $\|u\|_{y,2\rho} \leq c2^{2d}\|u\|_{y,\rho}$ (by (4.5)), we have by the $H^{2,p}$ regularity theory,

(2) $$|u_B|_{C^{1,1/2}(B_2(0))} \leq c^d, \qquad |\phi^B|_{C^{1,1/2}(B_2(0))} \leq c^d.$$

Notice that by (3.1)

(3) $$\mathscr{H}^{n-1}\{(\phi^B)^{-1}\{0\} \cap B_2(0)\} \leq cd, \qquad c = c(n).$$

Notice also that, for small enough ε, (1), (2), (3) and Lemma (4.8), with $\eta = (c\varepsilon)^{d/5}/2$, $w_1 = c^{-d}u_B$, $w_2 = c^{-d}\phi^B$, imply

$$\begin{aligned} &\mathscr{H}^{n-1}\{u_B^{-1}\{0\} \cap B_1 \cap \{|D\phi^B| > \eta\}\} \\ (4) \qquad &\leq \mathscr{H}^{n-1}\{u_B^{-1}\{0\} \cap B_1 \cap \{|Du_B| > \eta/2\}\} \\ &\leq 2\mathscr{H}^{n-1}\{(\phi^B)^{-1}\{0\} \cap B_2 \cap \{|D\phi^B| > \eta/4\}\} \leq cd. \end{aligned}$$

Also by (1)

$$(5) \qquad |Du_B|^{-1}\{0\}[\cap B_1 \cap \{|D\phi^B| > \eta\} = \varnothing.$$

Now we proceed to inductively define finite collections $\mathscr{S}_0, \mathscr{S}_1, \cdots$ of open balls, each collection covering $B_1(0) \cap |Du|^{-1}\{0\} \cap u^{-1}\{0\}|$ and having centers in $B_1(0) \cap u^{-1}\{0\}$ as follows:

$$\mathscr{S}_0 = \{B_1(0)\}.$$

Assume now that $l \geq 1$ and that $\mathscr{S}_0, \cdots, \mathscr{S}_{l-1}$ are already defined such that each ball in $\mathscr{S}_k$ has center in $B_1 \cap u^{-1}\{0\}$ and radius ε^{k-1}, and such that

$$(6) \qquad \cup_{B \in \mathscr{S}_k} B \supset B_1 \cap |Du|^{-1}\{0\} \cap u^{-1}\{0\},$$

$k = 0, 1, \cdots, l-1$. We now define $\mathscr{S}_l$. First for each $B = B_\rho(y) \in \mathscr{S}_{l-1}$ ($\rho = \varepsilon^{l-1}$), choose a harmonic polynomial ϕ^B of degree $\leq 3d$ as in (1), and let $\tau_B \colon \mathbf{R}^n \to \mathbf{R}^n$ be given by $\tau_B(x) = y + \rho x$. Cover $u^{-1}\{0\} \cap B \cap \tau_B\{|D\phi^B| < \eta\}$ ($\supset u^{-1}\{0\} \cap B \cap |Du|^{-1}\{0\}$ by (1)) with a collection $\mathscr{S}_l^B$ of balls with centers in $u^{-1}\{0\} \cap B_1(0)$ and radius $\varepsilon\rho$ ($= \varepsilon^l$) such that the balls of the same centers and $1/2$ the radius are pairwise disjoint. Then let $\mathscr{S}_l = \bigcup_{B \in \mathscr{S}_{l-1}} \mathscr{S}_l^B$. Notice that $\mathscr{S}_l$ covers $u^{-1}\{0\} \cap |Du|^{-1}\{0\} \cap B_1$ by construction, and hence the inductive definition of $\mathscr{S}_l$ is complete.

Since any pairwise disjoint collection of balls of radius ε with centers in $\{x \in B_1(0) \colon \operatorname{dist}(x, \{|D\phi_B| \leq \eta\}) < \varepsilon\}$ contains at most $cd^{2n+2}\varepsilon^{2-n}\log\varepsilon^{-1}$ balls by Theorem (3.3), we have

$$(7) \qquad \text{number of balls in } \mathscr{S}_l^B \leq cd^{2n+2}\varepsilon^{2-n}\log\varepsilon^{-1} \quad \forall B \in \mathscr{S}_{l-1}.$$

Let N_j denote the number of balls in the collection $\mathscr{S}_j$. Then the above inequality tells us that

$$N_j \leq N_{j-1} \cdot cd^{2n+2}\varepsilon^{2-n}\log\varepsilon^{-1} \quad \forall j \geq 1,$$

so by induction

$$(8) \qquad N_j \leq (cd^{2n+2}\varepsilon^{2-n}\log\varepsilon^{-1})^j \quad \forall j \geq 1.$$

Notice also that by (4) for each $B \in \mathscr{S}_{l-1}$ we have

$$\mathscr{H}^{n-1}\{u^{-1}\{0\} \cap B \cap \tau_B\{|D\phi_B| > \eta\}\} \leq cd\rho^{n-1},$$

where $\rho = \varepsilon^{l-1}$. Then since the collection $\mathscr{S}_l^B$ covers $u^{-1}\{0\} \cap B \cap \tau_B\{|D\phi^B| \leq \eta\}$, we get

$$(9) \qquad \mathscr{H}^{n-1}\left\{u^{-1}\{0\} \cap \left(\cup_{B\in\mathscr{S}_{l-1}} B\right) \sim \left(\cup_{B\in\mathscr{S}_l} B\right)\right\} \leq cdN_{l-1}(\varepsilon^{l-1})^{n-1}.$$

Since $B_1(0) \subset (\bigcup_l^\infty((\bigcup_{B\in\mathscr{S}_{l-1}} B) \sim (\bigcup_{B\in\mathscr{S}_l} B))) \cup (\bigcap_{l=0}^\infty(\bigcup_{j=l}^\infty \bigcup_{B\in\mathscr{S}_j} B))$ and since $\bigcap_{l=0}^\infty(\bigcup_{j=l}^\infty \bigcup_{B\in\mathscr{S}_j} B)$ is covered by $\bigcup_{j=l}^\infty \mathscr{S}_j$ for each l, we have by (9), (8) and the definition of $\mathscr{H}^{n-1}$ that

$$\mathscr{H}^{n-1}\{u^{-1}\{0\} \cap B_1\} \leq cd\sum_{l=1}^\infty (cd^{2n+2}\varepsilon \log \varepsilon^{-1})^{l-1} + \inf_{l\geq 1}\sum_{j=l}^\infty (cd^{2n+2}\varepsilon \log\varepsilon^{-1})^j \leq 2cd,$$

provided we take ε such that $cd^{2n+2}\varepsilon\log\varepsilon^{-1} \leq 1/2$. Notice that then we may take $\varepsilon = \varepsilon_0/d^{2n+3}$ for suitable $\varepsilon_0 = \varepsilon_0(n)$.

From (8) and the fact that each ball in the collection $\mathscr{S}_j$ has radius ε^j, we have

$$\sum_{B\in\mathscr{S}_j} (\operatorname{diam} B)^{n-2+\theta} \leq c(cd^{2n+2}\varepsilon^\theta \log\varepsilon^{-1})^j.$$

By choosing $cd^{2n+2}\varepsilon^\theta\log\varepsilon^{-1} \leq 1/2$ (note that this choice of ε—and hence the choices of ρ_0 for which the required hypothesis (4.4) holds–depends on θ), and letting $j \to \infty$, we conclude that $\mathscr{H}^{n-2-\theta}(B_1(0) \cap u^{-1}\{0\} \cap |Du|^{-1}\{0\}) = 0$ for each $\theta > 0$, because $\mathscr{S}_j$ covers $B_1(0) \cap u^{-1}\{0\} \cap |Du|^{-1}\{0\}$ for each $j \geq 1$ by construction.

5. Application to nodal sets of eigenfunctions

Consider a compact Riemannian manifold M with $C^{1,1}$ metric and let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ be the eigenvalues of Laplacian. Let ϕ_j be any eigenfunction corresponding to the eigenvalue λ_j.

According to the result of Donnelly and Fefferman [2, Theorem 4.2(ii)]

$$(5.1) \qquad \|\phi_j\|_{L^2(B_\rho(p))} \leq 2^{c\sqrt{\lambda_j}}\|\phi_j\|_{L^2(B_{\rho/2(p)})}$$

for $\rho \leq R$, where $c > 0$ depends only on an upper bound for the sectional curvatures and an upper bound for $\operatorname{diam} M$ and where R depends only on an upper bound for the sectional curvatures. Here $B_\rho(p)$ denotes the geodesic ball centered at p and having radius ρ. Of course the constant R can be selected so that we also have

$$(5.2) \qquad \left|\frac{\partial g_{ij}}{\partial x^k}\right| \leq 1, \qquad i, j, k = 1, \cdots, n,$$

where x denotes normal coordinates with origin corresponding to p.

(5.3) Theorem. *$\mathscr{H}^{n-1}(\phi_j^{-1}\{0\}) \le c\lambda_j^{c\sqrt{\lambda_j}}$ with c a constant depending only on an upper bound for the sectional curvatures of M and* diam M.

Remark. A much more precise upper bound is proved in the real analytic case in [2].

Proof of Theorem (5.3). In normal coordinates x, by (5.2) the equation $\Delta\phi_j = -\lambda_j\phi_j$ takes the form of equation (1.1) with $\sigma(t) \equiv t$, $\mu_1 = c(n)$, $\mu_2 = |\lambda_j|$. Then by (5.1) we have the hypotheses of Theorem (1.7) with $d = c\sqrt{\lambda_j}$ and with $\rho_0 = c\lambda_j^{-c\sqrt{\lambda_j}}$. Then Theorem (1.7) applies to give the bound

$$\mathscr{H}^{n-1}(B_\rho(p) \cap \phi_j^{-1}\{0\}) \le c\sqrt{\lambda_j}\rho^{n-1}$$

for $\rho \le c\lambda_j^{-c\sqrt{\lambda_j}}$ and any $p \in M$. Since we can cover M by a collection of $\le c \operatorname{vol} M/\rho^n$ such balls, we have the required result.

Appendix: Growth estimates

Here we record the growth results concerning elliptic equations which were needed for the present paper. (Somewhat more general, but less precise, estimates were introduced in [9], [10].)

We suppose

$$\text{(A.1)} \qquad \Delta u = a \cdot D^2 u + b \cdot Du + cu$$

on a ball $B_R(0) \subset \mathbf{R}^n$, and for $\rho \in (0, R)$ let

$$\text{(A.2)} \qquad \delta(\rho) \equiv \sup_{B_\rho(0)} |a| + \rho \sup_{B_\rho(0)} |b| + \rho^2 \sup_{B_\rho(0)} |c|.$$

Then we have:

(A.3) **Theorem.** *For any given $\theta \in (0,1)$ and any $q \in [1/2, \infty) \sim \{1, 2, \cdots\}$, there exists $\varepsilon = \varepsilon(n, \theta, \operatorname{dist}(q, \{1, 2, \cdots\}))$ such that if* (A.1) *holds, and $\rho_0 \in (0, R)$ is such that $\delta(\rho_0) \le \varepsilon^q$ ($\delta(\rho)$ as in* (A.2)*), then for any $\rho \in (0, \rho_0]$*

$$\|u\| \ge \theta^q \|u\|_\rho \Rightarrow \|u\|_{\theta^2\rho} \ge \theta^q \|u\|,$$

where we use the notation $\|u\|_\rho^2 = \rho^{-n}\int_{B_\rho(0)} u^2$.

Proof. First note that if $\delta(\rho) \equiv 0$, then u is a harmonic function and in this case (using expansion by harmonic polynomials) we have

$$\text{(1)} \qquad \|u\|_\rho^2 = \sum_{k=0}^{\infty} a_k^2 \rho^{2k}, \qquad \rho \in (0, R),$$

for suitable constants a_k. In this case the lemma follows immediately because if $f(\rho)$ denotes the series with nonnegative coefficients on the right of (1) then $\log f(e^{-t})$ is a convex decreasing function of t, as one checks by direct computation. (We emphasize that this latter fact is true simply because the coefficients in the power series expansion of f are nonnegative and has nothing intrinsically to do with harmonic functions.) This in particular gives

$$(2) \quad \left(\frac{f(\rho_2)}{f(\rho_3)}\right)^{1/\log(\rho_3/\rho_2)} \leq \left(\frac{f(\rho_1)}{f(\rho_2)}\right)^{1/\log(\rho_2/\rho_1)}, \quad 0 < \rho_1 < \rho_2 < \rho_3 < R,$$

and equality for some $\rho_1 < \rho_2 < \rho_3 < R$ implies that

$$\left(\frac{f(\sigma_1)}{f(\sigma_2)}\right)^{1/\log(\sigma_2/\sigma_1)} \equiv \text{const}, \qquad 0 < \sigma_1 < \sigma_2 < \rho_3,$$

which means $f(\rho) \equiv \text{const}\, \rho^l$ for some constant l; l must be an even integer by (1)).

Also by an obvious compactness argument (using (2) and the normalizations $f(\rho_2) = 1, \rho_2 = 1$) we deduce that for any given $\alpha_2 > \alpha_1 > 1$ and any given $q \geq 1/2, q \notin \{1, 2, \cdots\}$ $\exists \varepsilon = \varepsilon(\alpha_1, \alpha_2, \text{dist}(q, \{1, 2, \cdots\})) > 0$ such that for any $0 < \rho_1 < \rho_2 < \rho_3 < R$ with $\alpha_1 \leq \rho_2/\rho_1 \leq \alpha_2$ and $\alpha_1 \leq \rho_3/\rho_2 \leq \alpha_2$ we have

$$(3) \qquad \frac{f(\rho_2)}{f(\rho_3)} \geq \left(\frac{\rho_2}{\rho_3}\right)^q \Rightarrow \frac{f(\rho_1)}{f(\rho_2)} \geq \left(\frac{\rho_1}{\rho_2}\right)^{q-\varepsilon}.$$

In the general case $\delta(\rho) \leq \varepsilon^q$, we approximate by the solution v of the equation for $\delta = 0$, having the same boundary values as u on $\partial B_{\alpha^{-1}\rho}(0)$, with $\alpha > 1$ suitably close to 1. According to the $H^{2,p}$ regularity theory we then have

$$\|u - v\|_{\alpha^{-1}\rho} \leq c\delta(\rho)(\alpha - 1)^{-c}\|u\|_\rho$$

with $c = c(n)$. Thus using this together with (3) we easily conclude the required result (A.3).

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University of Minnesota
Stanford University